

# Multiple Iterative Splitting method for Higher order and Integro-differential equations

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**Abstract.** In this paper we present an extension of standard iterative splitting schemes to multiple splitting schemes for solving higher order differential equations.

We are motivated by dynamical systems, which occur in dynamics of the electrons in the plasma using a simplified Boltzmann equation. Oscillation problems in spectroscopy problems using wave-equations.

The motivation arose to simulate active plasma resonance spectroscopy which is used for plasma diagnostic techniques, see [2], [16] and [18].

**Keywords:** kinetic model, neutron transport, dynamics of electrons, transport equation, splitting schemes, semi-group.

**AMS subject classifications.** 35K25, 35K20, 74S10, 70G65.

## 1 Introduction

We motivate our studying on simulating a active plasma resonance spectroscopy which is well established in plasma diagnostic techniques.

To study the model with simulation models, we concentrate on an abstract kinetic model, which described the dynamics of electrons in the plasma by using a Boltzmann equation. The Boltzmann equation is coupled with the electric field and we obtain coupled partial differential equations.

The paper is outlined as follows.

In section 2 we present our mathematical model and a possible reduced model for the further approximations.

The functional analytical setting with the higher order differential equations are discussed in section 3.

The splitting schemes are presented in in Section 4.

Numerical experiments are done in Section 5. In the contents, that are given in Section 6, we summarize our results.

## 2 Mathematical Model

In the following a model is presented due to the motivation in [2], [16] and [18].

The models consider a fluid dynamical approach of the natural ability of plasmas to resonate in the near of the electron plasma frequency  $\omega_{pe}$ .

Here we specialize to an abstract kinetic model to describe the dynamics of the electrons in the plasma, that allows to do the resonance.

The Boltzmann equation for the electron particles are given as

$$\begin{aligned} \frac{\partial f(x, v, t)}{\partial t} &= -v \cdot \nabla_x f(x, v, t) - \frac{e}{m_e} \nabla_x \phi \cdot \nabla_v f(x, v, t) \\ &\quad - \sigma(x, v, t) f(x, v, t) + \int_V \kappa(x, v, v') f(x, v', t) dv', \end{aligned} \quad (1)$$

$$f(x, v, 0) = f_0(x, v), \quad (2)$$

and boundary conditions are postulated at the boundaries of  $P$  (plasma).

In front of the materials we assume complete reflection of the electrons due to the sheath  $f(v_{||} + v_{\perp})$  with  $v_{||}$  is the parallel and  $v_{\perp}$  perpendicular to the surface normal vector.  $\phi$  is the electric field.

## 3 Higher order differential equations

We consider the abstract homogeneous Cauchy problem in a Banach space  $\mathbf{X} \in \mathbb{R}^n$ :

$$A_0 \frac{d^n \mathbf{u}(t)}{dt^n} + A_1 \frac{d^{n-1} \mathbf{u}(t)}{dt^{n-1}} + \dots + A_n = \mathbf{f}(t), \quad (3)$$

$$\frac{d^{i-1} \mathbf{u}(t)}{dt^{i-1}} = \mathbf{u}_{i-1}, i = 1, \dots, n, \quad (4)$$

where  $A_0, \dots, A_n \in \mathbf{X} \times \mathbf{X}$  are bounded operators and  $\|\cdot\|$  is the corresponding norm in  $\mathbf{X}$  and let  $\|\cdot\|_{L(\mathbf{X})}$  be the induced operator norm.

For the transformation we have the following assumptions:

**Assumption 31** 1.) The function  $f(t)$  is given as:

$$\mathbf{f}(t) = 0, \quad (5)$$

otherwise we solve a non-autonomous equation.

2.) We assume that the characteristic polynomial:

$$A_0 \lambda^n + A_1 \lambda^{n-1} + \dots + A_n = 0, \quad (6)$$

has solution of complex valued matrices in  $\mathbf{X} \times \mathbf{X} \in \mathbb{C}^m \times \mathbb{C}^m$ , given as:

$$(\lambda I - B_1)(\lambda I - B_2) + \dots + (\lambda I - B_n) = 0, \quad (7)$$

**Corollary 1.** *The higher order differential equation (3) can be decoupled with the assumptions 31 to the following differential equation:*

$$\frac{d\mathbf{u}_1(t)}{dt} - B_1\mathbf{u}_1 = 0, \quad (8)$$

$$\frac{d\mathbf{u}_2(t)}{dt} - B_2\mathbf{u}_2 = 0, \quad (9)$$

$$\dots \quad (10)$$

$$\frac{d\mathbf{u}_n(t)}{dt} - B_n\mathbf{u}_n = 0, \quad (11)$$

where the analytical solution is given as:

$$\mathbf{u}(t) = \sum_{i=1}^n d_i \mathbf{u}_{i,0} = \sum_{i=1}^n \exp(B_i t) d_i \mathbf{u}_{i,0}. \quad (12)$$

and  $d_i$  are given via the initial conditions.

*Proof.* The solutions can be derived via the characteristics polynomial (idea of scalar linear differential equations) and the idea of the superposition of the linear combined solutions.

*Remark 1.* The initial conditions are computed by solving the Vandermode matrix, see the ideas in [17].

We have to solve:

$$\begin{pmatrix} I & I & \dots & I \\ B_1 & B_2 & \dots & B_n \\ B_1^2 & B_2^2 & \dots & B_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ B_1^{n-1} & B_2^{n-1} & \dots & B_n^{n-1} \end{pmatrix} \cdot \begin{pmatrix} d_1 u_{1,0} \\ d_2 u_{2,0} \\ d_3 u_{3,0} \\ \vdots \\ d_n u_{n,0} \end{pmatrix} = \begin{pmatrix} u(0) \\ \frac{\partial}{\partial t} u(0) \\ \frac{\partial^2}{\partial t^2} u(0) \\ \vdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}} u(0) \end{pmatrix} \quad (13)$$

A further simplification can be done to rewrite the integral-differential equation in two first order differential equations. Later such a reduction allows us to apply fast iterative splitting methods.

**Corollary 2.** *The higher order differential equation (3) can be transformed with the assumptions 31 to two first order differential equation:*

$$\frac{d\mathbf{u}_1(t)}{dt} = B_1\mathbf{u}_1(t) \quad (14)$$

$$\mathbf{u}_1(0) = d_1\mathbf{u}_{1,0}, \quad (15)$$

$$\dots \quad (16)$$

$$\frac{d\mathbf{u}_n(t)}{dt} = B_n\mathbf{u}_n(t) \quad (17)$$

$$\mathbf{u}_n(0) = d_n\mathbf{u}_{n,0}, \quad (18)$$

where we have  $B_i = B_{i1} + B_{i2}$  for  $i = 1 \dots, n$ .

The analytical solution are given as:

$$\mathbf{u}(t) = \sum_{i=1}^n \exp(B_i t) d_i \mathbf{u}_{i,0}.$$

*Proof.* The analytical solution of the first order differential equation (14) and (17) are given by each characteristic polynomial:

$$\lambda_1 I - (B_{11} + B_{12}) = 0, \quad (19)$$

$$\dots \quad (20)$$

$$\lambda_n I - (B_{n1} + B_{n2}) = 0, \quad (21)$$

while the solution is given as with the notations:

$$\lambda_i I = B_{i1} + B_{i2}, \quad i = 1, \dots, n, \quad (22)$$

and therefore the analytical solution is given as (19).

Therefore this is the solution of our integro-differential equation (3) with the assumptions 31.

## 4 Splitting schemes

The operator-splitting methods are used to solve complex models in the geophysical and environmental physics, they are developed and applied in [22], [23] and [24]. This ideas based in this article are solving simpler equations with respect to receive higher order discretization methods for the remain equations. For this aim we use the operator-splitting method and decouple the equation as follows described.

In the following we concentrate on the iterative-splitting method.

### 4.1 Iterative splitting method for Integro-differential equations

The following algorithm is based on the iteration with fixed splitting discretization step-size  $\tau$ , namely, on the time interval  $[t^n, t^{n+1}]$  we solve the following sub-problems consecutively for  $i = 0, 2, \dots, 2m$ . (Cf. [15] and [11].)

$$\frac{\partial c_{ij}(t)}{\partial t} = B_{i1} c_{ij}(t) + B_{i2} c_{ij-1}(t), \text{ with } c_{ij}(t^n) = d_i c^n \quad (23)$$

$$\text{and } c_{i0}(t^n) = c^n, \quad c_{i,-1} = 0.0,$$

$$\frac{\partial c_{i,j+1}(t)}{\partial t} = B_{i1} c_{ij}(t) + B_{i2} c_{i,j+1}(t), \quad (24)$$

$$\text{with } c_{i,j+1}(t^n) = d_i c^n,$$

where  $c^n$  is the known split approximation at the time level  $t = t^n$ . The split approximation at the time-level  $t = t^{n+1}$  is defined as  $c^{n+1} = \sum_{k=1}^n c_{k,2m+1}(t^{n+1})$ .

**Theorem 1.** *Let us consider the abstract Cauchy problem in a Banach space  $\mathbf{X}$*

$$\begin{aligned} \partial_t c(t) &= B_{i1}c(t) + B_{i2}c(t), \quad 0 < t \leq T \\ c(0) &= d_i c_{i,0} \end{aligned} \quad i = 1, \dots, n, \quad (25)$$

where  $B_{i1}, B_{i2}, B_{i1} + B_{i2} : \mathbf{X} \rightarrow \mathbf{X}$  are given linear operators being generators of the  $C_0$ -semi-group and  $c_{i,0} \in \mathbf{X}$  is a given element. Then the iteration process (31)–(32) is convergent and the rate of the convergence is of second order.

*Proof.* The proof is done in the work of Geiser [14].

The algorithm is given as:

**Algorithm 41**

$$\frac{\partial c^i(t)}{\partial t} = A c^i(t) + B c^{i-1}(t), \text{ with } c^i(t^n) = c^{i-1}(t^{n+1}) \quad (26)$$

and the starting values  $c^0(t^n) = c(t^n)$  results of last iteration,  $c^{-1}(t^n) = 0.0$ ,

$$\frac{\partial c^{i+1}(t)}{\partial t} = A c^i(t) + B c^{i+1}(t), \quad (27)$$

with  $c^{i+1}(t^n) = c^i(t^{n+1})$ ,

$$\epsilon > |c^{i+1}(t^{n+1}) - c^{i-1}(t^{n+1})| \text{ Stop criterion} \quad (28)$$

$$\text{result for the next time-step} \quad (29)$$

$$c(t^{n+1}) = c^m(t^{n+1}), \text{ for } m \text{ fulfill the stop-criterion} \quad (30)$$

for each  $i = 0, 2, \dots$ , where  $c^n$  is the known split approximation at the previous time level.

In the following we concentrate on the iterative-splitting method.

## 4.2 Iterative splitting method for higher order differential equations

The following algorithm is based on the iteration with fixed splitting discretization step-size  $\tau$ , namely, on the time interval  $[t^n, t^{n+1}]$  we solve the following sub-problems consecutively for  $j = 0, 2, \dots, 2m$ . (Cf. [15] and [11].)

$$\frac{\partial c_{ij}(t)}{\partial t} = B_{i1}c_{ij}(t) + B_{i2}c_{i,j-1}(t), \text{ with } c_{ij}(t^n) = d_i c^n \quad (31)$$

and  $c_{i0}(t^n) = c^n$ ,  $c_{i,-1} = 0.0$ ,

$$\frac{\partial c_{i,j+1}(t)}{\partial t} = B_{i1}c_{ij}(t) + B_{i2}c_{i,j+1}(t), \quad (32)$$

with  $c_{i,j+1}(t^n) = d_i c^n$ ,

where  $i = 1, \dots, I$  are the number of equations. Further  $c^n$  is the known split approximation at the time level  $t = t^n$ . The split approximation at the time-level  $t = t^{n+1}$  is defined as  $c^{n+1} = \sum_{k=1}^n c_{k,2m+1}(t^{n+1})$ .

**Theorem 2.** *Let us consider the abstract Cauchy problem in a Banach space  $\mathbf{X} \subset \mathbb{C}$ :*

$$\begin{aligned} \partial_t c(t) &= B_{i1}c(t) + B_{i2}c(t), \quad 0 < t \leq T \\ c(0) &= d_i c_0 \end{aligned} \quad i = 1, \dots, n, \quad (33)$$

where  $d_i \in \mathbb{C}$  is the constant based on the initial conditions, further  $B_{i1}, B_{i2}, B_{i1} + B_{i2} : \mathbf{X} \rightarrow \mathbf{X}$  are given linear operators being generators of the  $C_0$ -semi-group and  $c_0 \in \mathbf{X}$  is a given element. Then the iteration process (31)–(32) is convergent and the rate of the convergence is of second order.

*Proof.* The proof is done in the work of Geiser [14].

The algorithm is given as:

**Algorithm 42**

$$\frac{\partial c_{i,j}(t)}{\partial t} = B_{1,i}c_{i,j}(t) + B_{2,i}c_{i-1,j}(t), \text{ with } c_i(t^n) = c^{i-1}(t^{n+1}) \quad (34)$$

and the starting values  $c_{i,0}(t^n) = c_i(t^n)$  results of last iteration,  $c_{i,-1}(t^n) = 0.0$ ,

$$\frac{\partial c_{i,j+1}(t)}{\partial t} = B_{1,i}c_{i,j}(t) + B_{2,i}c_{i,j+1}(t), \quad (35)$$

with  $c^{i+1}(t^n) = c^i(t^{n+1})$ ,

$$\epsilon > |c^{i+1}(t^{n+1}) - c^{i-1}(t^{n+1})| \text{ Stop criterion} \quad (36)$$

result for the next time-step (37)

$$c(t^{n+1}) = c^m(t^{n+1}), \text{ for } m \text{ fulfill the stop-criterion} \quad (38)$$

for each  $j = 0, 2, \dots$ , where  $c^n$  is the known split approximation at the previous time level.

Further  $B_i = B_{1,i} + B_{2,i}$  is a decomposition of the matrix  $B_i$ .

We reformulate to an algorithm that deals only with real numbers and rewrite:

$$\begin{aligned} \partial_t (c_{re}(t) + ic_{im}(t)) &= (B_{re,i1} + iB_{im,i1})(c_{re}(t) + ic_{im}(t)) \\ &+ (B_{re,i2} + iB_{im,i2})(c_{re}(t) + ic_{im}(t)), \quad 0 < t \leq T, \\ (c_{re}(0) + ic_{im}(0)) &= (d_{i,re} + id_{i,im})(c_{re,0} + ic_{im,0}) \quad i = 1, \dots, n, \end{aligned} \quad (39)$$

We have the following algorithm:

**Algorithm 43** *In the following, we have two iteration processes:*

- First iteration process:  $j = 1, \dots, J$  iterates over the decomposition of the matrices  $B_1, B_2$ .
- Second iteration process:  $k = 1, \dots, K$  iterates over the real and imaginary parts.

We start with  $j = 0, k = 0$ ,  
First we iterate over  $j$

$$\begin{aligned} \frac{\partial(c_{i,re}^{j,k})(t)}{\partial t} &= B_{1,re}c_{i,re}^{j,k}(t) + B_{2,re}c_{i,re}^{j-1,k}(t) \\ &- B_{1,im}c_{i,im}^{j-1,k-1}(t) - B_{2,im}c_{i,im}^{j-1,k-1}(t), \\ \frac{\partial(c_{i,im}^{j,k})(t)}{\partial t} &= B_{1,re}c_{i,im}^{j,k}(t) + B_{2,re}c_{i,im}^{j-1,k}(t) \\ &+ B_{1,im}c_{i,re}^{j-1,k-1}(t) + B_{2,im}c_{i,re}^{j-1,k-1}(t), \\ \text{with the initial condition } c_{i,re}^{j,k}(t^n) &= c_{i,re}^{j,k,n}, c_{i,im}^{j,k}(t^n) = c_{i,im}^{j,k,n} \\ \text{with the starting condition } c_{i,re}^{-1,k}(t^n) &= 0, c_{i,im}^{-1,k}(t^n) = 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial(c_{i,re}^{j+1,k})(t)}{\partial t} &= B_{1,re}c_{i,re}^{j,k}(t) + B_{2,re}c_{i,re}^{j+1,k}(t) \\ &- B_{1,im}c_{i,im}^{j-1,k-1}(t) - B_{2,im}c_{i,im}^{j-1,k-1}(t), \\ \frac{\partial(c_{i,im}^{j+1,k})(t)}{\partial t} &= B_{1,re}c_{i,im}^{j,k}(t) + B_{2,re}c_{i,im}^{j+1,k}(t) \\ &+ B_{1,im}c_{i,re}^{j-1,k-1}(t) + B_{2,im}c_{i,re}^{j-1,k-1}(t), \\ \text{with the initial condition } c_{i,re}^{j+1,k}(t^n) &= c_{i,re}^{j+1,k,n}, c_{i,im}^{j+1,k}(t^n) = c_{i,im}^{j+1,k,n} \end{aligned}$$

if  $j = J$  or the iteration error over  $j$  is less err then we iterate over  $k$ .  
Further  $c^n$  is the known split approximation at the time level  $t = t^n$ , cf. [7].  
Further  $B_i = B_{1,i} + B_{2,i}$  is a decomposition of the matrix  $B_i$ .

## 5 Experiments for the Plasma resonance spectroscopy

In the following, we present different examples.

### 5.1 First Example: Matrix problem with integral term

We deal with a simpler integro-differential equations:

$$\frac{dc}{dt} - Ac(t)dt = B \int c(t') dt', \quad t \in [0, 1], \quad (40)$$

and the transformed second order differential equation is given as:

$$\partial_{tt}c = A\partial_t c + Bc \quad (41)$$

$$(42)$$

and the operators for the splitting scheme are given as:

$$\tilde{A} = -\frac{A}{2}, \tilde{B} = \sqrt{\frac{A A^T}{4}} - B \quad (43)$$

while  $\tilde{A}^T$  is the transposed matrix of  $\tilde{A}$ .

The matrices are given as

$$A = \begin{pmatrix} -0.01 & 0.01 & 0 & \dots & & & & & & & \\ 0.01 & -0.01 & 0 & \dots & & & & & & & \\ 0.01 & 0.01 & -0.02 & 0 & \dots & & & & & & \\ 0.01 & 0.01 & 0.01 & -0.03 & 0 & \dots & & & & & \\ \vdots & & & & & & & & & & \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & -0.08 & 0 & \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0 & -0.08 & \end{pmatrix}, \quad (44)$$

$$B = \begin{pmatrix} -0.08 & 0 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\ 0 & -0.08 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\ \vdots & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.02 & 0.01 & 0.01 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 & 0.01 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & -0.01 & \end{pmatrix}. \quad (45)$$

The Figure 1 present the numerical errors between the exact and the numerical solution. Here we obtain results for one-side and two-side iterative schemes on operators  $A$  and  $B$ .

The computational results are given in the Figure 2, we present the one-side and two-side iterative results.

The Figure 3 present the numerical errors between the exact and the numerical solution for the optimized iterative schemes. Here we obtain results for one-side and two-side iterative schemes on operators  $A$  and  $B$ .

*Remark 2.* For the computations, we see the benefit of the optimal iterative schemes, which applied the two iterative steps of the two solutions in one scheme, see Section 4. The best results are given by the one-side iterative scheme with respect to the operator  $B$ .

## 5.2 Second Example: Third order differential equations

We deal with a simple third order differential equations:

$$\frac{d^3 c}{dt^3} - Ac(t) = 0, \quad t \in [0, 1], \quad (46)$$

$$c(0) = (1, \dots, 1)^t \in \mathbb{C}^m, \quad (47)$$

$$c'(0) = \frac{1 - \sqrt{2}}{3} A^{1/3} c(0), \quad (48)$$

$$c''(0) = \frac{1}{3} A^{2/3} c(0), \quad (49)$$



$A \in \mathbb{C}^m \times \mathbb{C}^m$ ,  $c : \mathbb{R}^+ \rightarrow \mathbb{C}^m$  is sufficient smooth ( $c \in C^3(\mathbb{R}^+)$ ) and we have  $m = 10$ .

The transformed first order differential equations are given as:

$$\partial_t c_1 - A^{1/3} c_1 = 0 \quad (50)$$

$$\partial_t c_2 - A^{1/3} \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) c_3 = 0 \quad (51)$$

$$\partial_t c_3 - A^{1/3} \left( -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) c_3 = 0 \quad (52)$$

$$(53)$$

where  $c = \sum_{i=1}^3 d_i c_i(t)$  and  $d_1, \dots, d_3$  are given with respect to the initial conditions and are given as  $d_1 = d_2 = d_3 = \frac{1}{3}c(0)$ .

Further the operators for the splitting scheme for the three iterative splitting schemes are given as:

$$A_{1,1,re} = \text{diag}(A^{1/3}), A_{1,2,re} \text{outerdiag}(A^{1/3}), \quad (54)$$

$$A_{2,1,re} = -\frac{\sqrt{2}}{2} \text{diag}(A^{1/3}), A_{2,2,re} = -\frac{\sqrt{2}}{2} \text{outerdiag}(A^{1/3}), \quad (55)$$

$$A_{2,1,im} = \frac{\sqrt{2}}{2} \text{diag}(A^{1/3}), A_{2,2,im} = \frac{\sqrt{2}}{2} \text{outerdiag}(A^{1/3}), \quad (56)$$

$$A_{3,1,re} = -\frac{\sqrt{2}}{2} \text{diag}(A^{1/3}), A_{3,2,re} = -\frac{\sqrt{2}}{2} \text{outerdiag}(A^{1/3}), \quad (57)$$

$$A_{3,1,im} = -\frac{\sqrt{2}}{2} \text{diag}(A^{1/3}), A_{3,2,im} = -\frac{\sqrt{2}}{2} \text{outerdiag}(A^{1/3}), \quad (58)$$

The matrix  $A$  is given as

$$A = \begin{pmatrix} -0.01 & 0.01 & 0 & \dots & & & & & & \\ 0.01 & -0.01 & 0 & \dots & & & & & & \\ 0.01 & 0.01 & -0.02 & 0 & \dots & & & & & \\ 0.01 & 0.01 & 0.01 & -0.03 & 0 & \dots & & & & \\ \vdots & & & & & & & & & \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & -0.08 & 0 \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0 & -0.08 \end{pmatrix}, \quad (59)$$

Here, we deal with the following splitting schemes:

- $c_1$  is computed by a scalar iterative scheme.
- $c_2, c_3$  are computed by a vectorial iterative scheme (because of real and imaginary parts).

For  $c_1$  we have:

$$\frac{\partial \mathbf{C}_1^i(t)}{\partial t} = \mathcal{A}_{11} \mathbf{C}_1^i(t) + \mathcal{A}_{12} \mathbf{C}_1^{i-1}(t), \text{ with } \mathbf{C}_1^i(t^n) = \mathbf{C}_1^{i-1}(t^{n+1}) \quad (60)$$

$$\text{and the starting values } \mathbf{C}_1^0(t^n) = \frac{1}{3} \mathbf{C}(t^n) \quad (61)$$

where  $\mathbf{C}_1 = (c_{1,re} + ic_{1,im})^t$  and

$$\mathcal{A}_{11} = \begin{pmatrix} A_{1,1,re} & 0 \\ 0 & A_{1,1,im} \end{pmatrix}, \mathcal{A}_{12} = \begin{pmatrix} A_{1,2,re} & 0 \\ 0 & A_{1,2,im} \end{pmatrix}, \quad (62)$$

for  $i = 1, 2, \dots, I$  and the solution is given as  $\mathbf{C}_1^i(t^{n+1})$ .

For  $c_2$  we have:

$$\frac{\partial \mathbf{C}_2^i(t)}{\partial t} = \mathcal{A}_{21} \mathbf{C}_2^i(t) + \mathcal{A}_{22} \mathbf{C}_2^{i-1}(t), \text{ with } \mathbf{C}_2^i(t^n) = \mathbf{C}_2^{i-1}(t^{n+1}) \quad (63)$$

$$\text{and the starting values } \mathbf{C}_2^0(t^n) = \frac{1}{3} \mathbf{C}(t^n) \quad (64)$$

where  $\mathbf{C}_2 = (c_{2,re} + ic_{2,im})^t$  and

$$\mathcal{A}_{21} = \begin{pmatrix} A_{2,1,re} & 0 \\ 0 & A_{2,1,im} \end{pmatrix}, \mathcal{A}_{22} = \begin{pmatrix} A_{2,2,re} & -(A_{2,1,im} + A_{2,2,im}) \\ (A_{2,1,im} + A_{2,2,im}) & A_{2,2,re} \end{pmatrix} \quad (65)$$

for  $i = 1, 2, \dots, I$  and the solution is given as  $\mathbf{C}_2^i(t^{n+1})$ .

For  $c_3$  we have:

$$\frac{\partial \mathbf{C}_3^i(t)}{\partial t} = \mathcal{A}_{31} \mathbf{C}_3^i(t) + \mathcal{A}_{32} \mathbf{C}_3^{i-1}(t), \text{ with } \mathbf{C}_3^i(t^n) = \mathbf{C}_3^{i-1}(t^{n+1}) \quad (66)$$

$$\text{and the starting values } \mathbf{C}_3^0(t^n) = \frac{1}{3} \mathbf{C}(t^n) \quad (67)$$

where  $\mathbf{C}_3 = (c_{3,re} + ic_{3,im})^t$  and

$$\mathcal{A}_{31} = \begin{pmatrix} A_{3,1,re} & 0 \\ 0 & A_{3,1,im} \end{pmatrix}, \mathcal{A}_{32} = \begin{pmatrix} A_{3,2,re} & -(A_{3,1,im} + A_{3,2,im}) \\ (A_{3,1,im} + A_{3,2,im}) & A_{3,2,re} \end{pmatrix} \quad (68)$$

for  $i = 1, 2, \dots, I$  and the solution is given as  $\mathbf{C}_3^i(t^{n+1})$ .

The solution is given as:

$$\mathbf{C}^i(t^{n+1}) = \sum_{j=1}^3 \mathbf{C}_j^i(t^{n+1}).$$

The computational results for the optimized iterative schemes are given in the Figure 4, we present the one-side and two-side iterative results.

*Remark 3.* For the computations, we see the benefit of the optimal iterative schemes. While we deal with real and imaginary parts, it is important to reduce the computational costs. We applied in one scheme the real and imaginary solution, see Section 4. The best results are given by the one-side iterative scheme with respect to the operator  $B$ .

## 6 Conclusions and Discussions

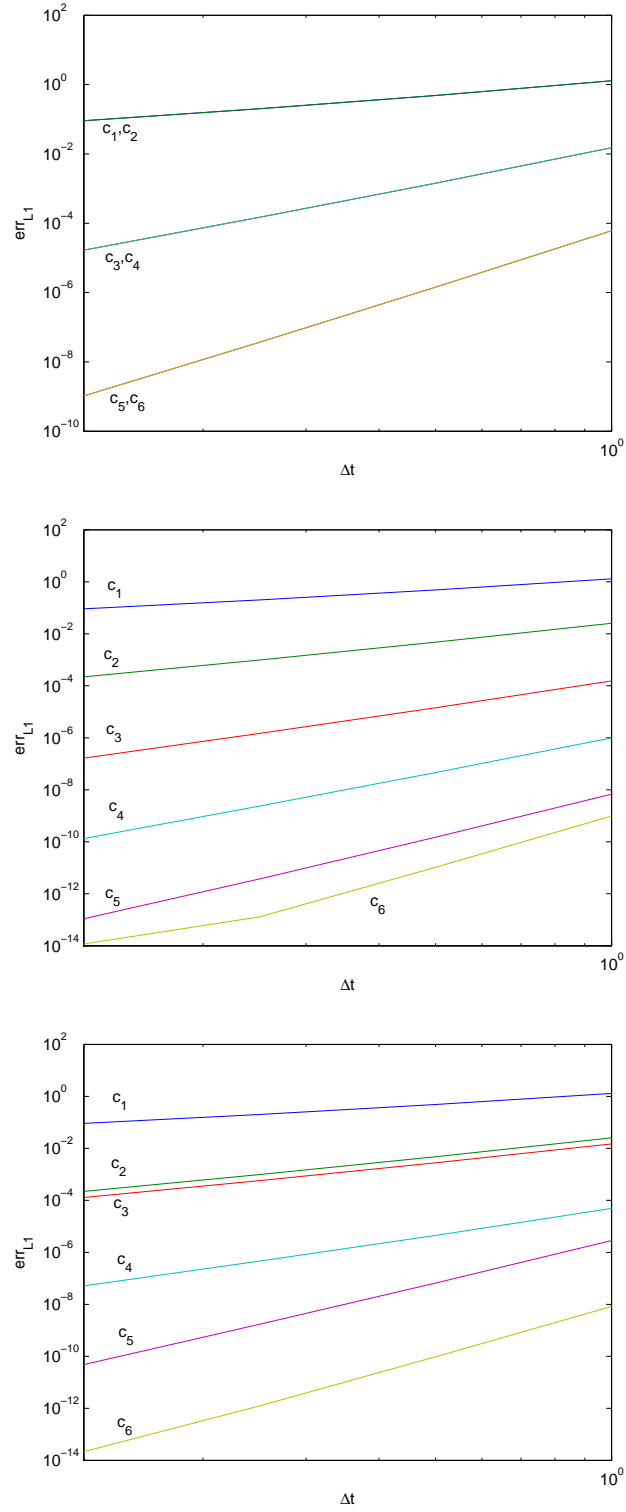
We present the coupled model for a transport model for deposition species in a plasma environment. We assume the flow field is computed by the plasma model and the transport of the deposition species with a transport-reaction model.

Based on the physical effects, we deal with higher order differential equations (scattering parts, reaction parts, etc.). We validate a novel splitting schemes, that embedded the real and imaginary parts of the solutions. Standard iterative splitting schemes can be extended to such complex iterative splitting schemes. First computations help to understand the important modeling of the plasma environment in a CVD reactor with scattering and higher order time-derivative parts. In future, we work on a general theory of embedding the complex schemes to standard splitting schemes.

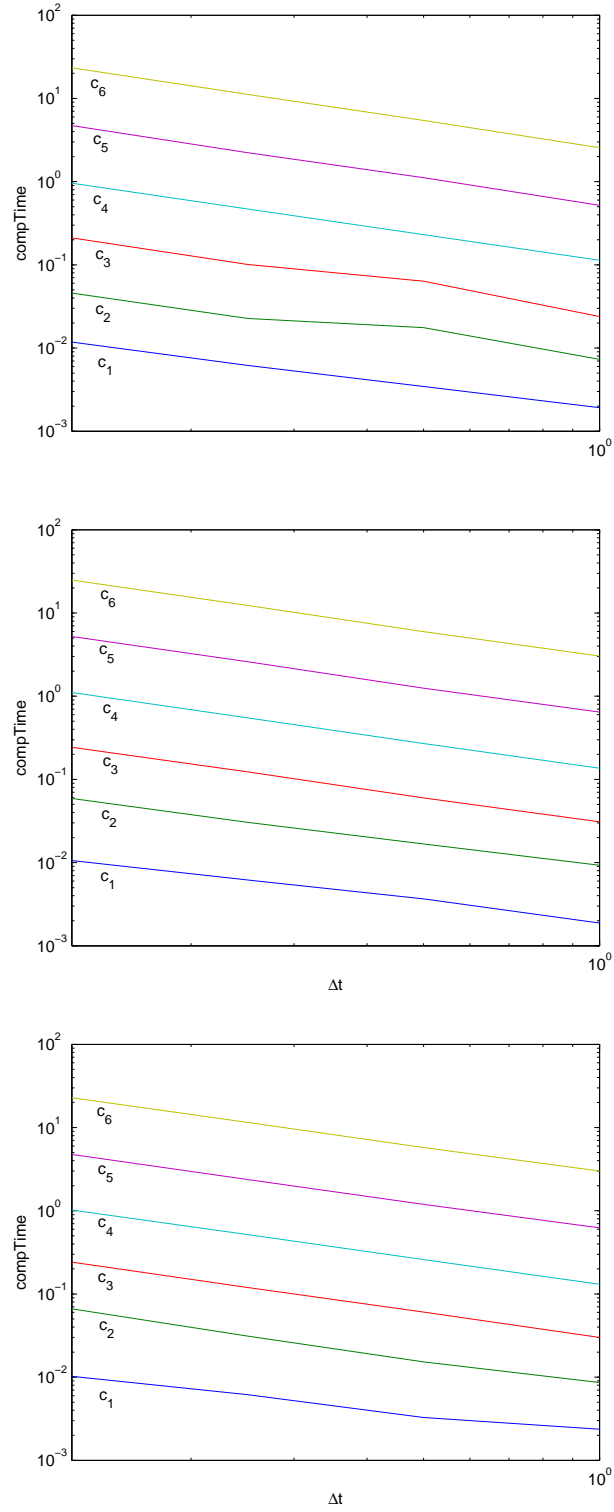
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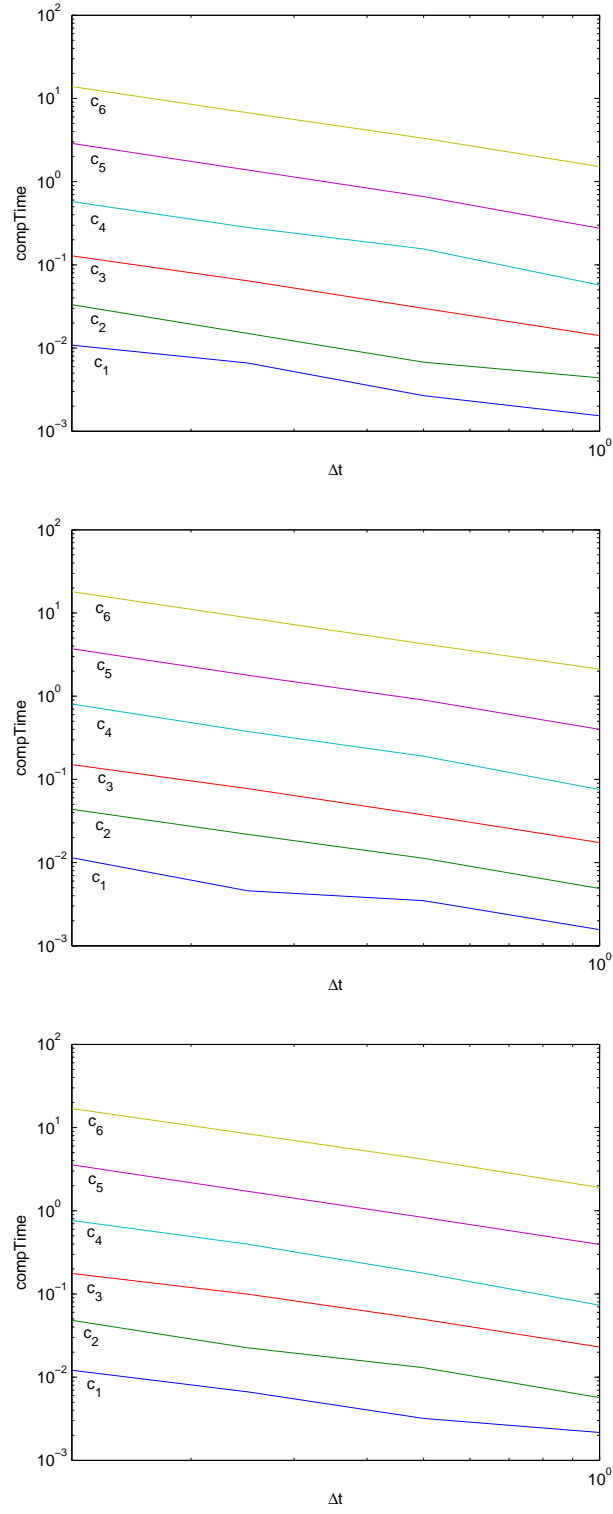
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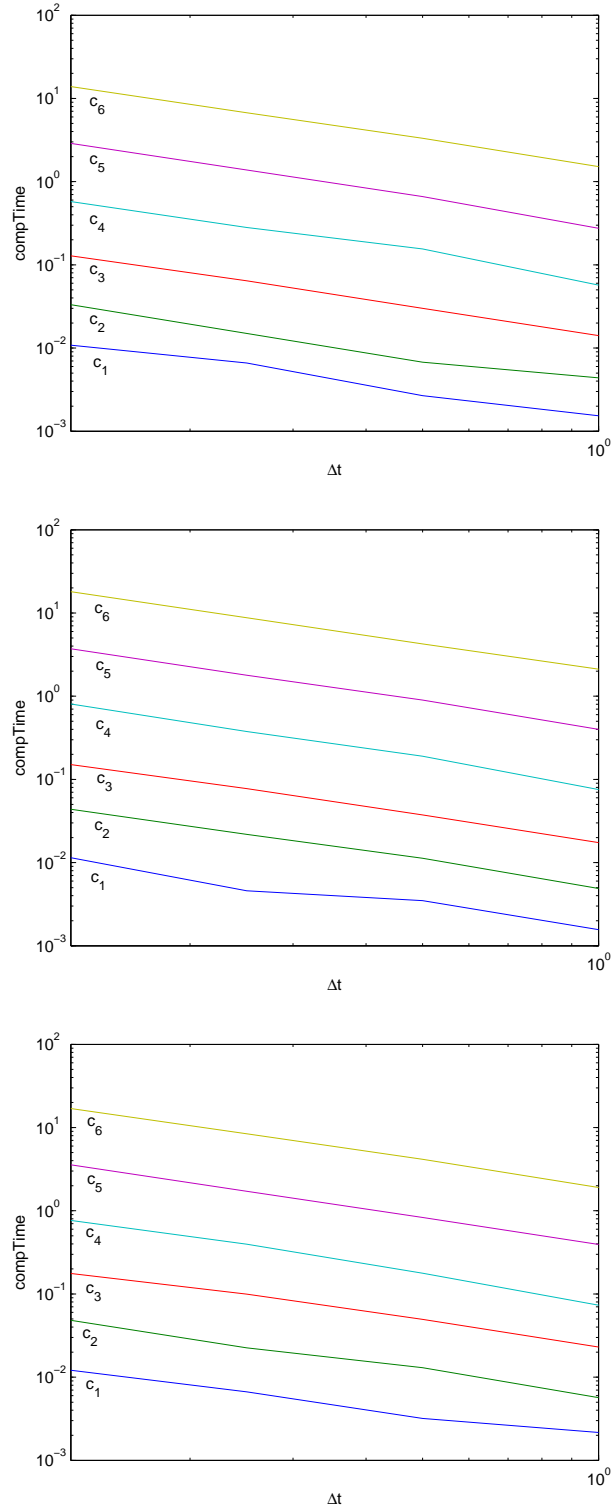
**Fig. 1.** Numerical errors of the one-side Splitting scheme with  $A$  (upper figure), the one-side Splitting scheme with  $B$  (middle figure) and the two-side iterative schemes with  $1, \dots, 6$  iterative steps (lower figure).



**Fig. 2.** The computational time of the one-side and two-side Splitting scheme: one-side splitting over  $A$  (upper figure), one-side splitting over  $B$  (middle figure) and two-side splitting scheme alternating between  $A$  and  $B$  (lower figure) with  $1, \dots, 6$  iterative steps.



**Fig. 3.** Numerical errors of the one-side Splitting scheme with  $A$  (upper figure), the one-side Splitting scheme with  $B$  (middle figure) and the two-side iterative schemes with  $1, \dots, 6$  iterative steps (lower figure).



**Fig. 4.** The computational time of the one-side and two-side Splitting scheme: one-side splitting over  $A$  (upper figure), one-side splitting over  $B$  (middle figure) and two-side splitting scheme alternating between  $A$  and  $B$  (lower figure) with  $1, \dots, 6$  iterative steps.



